Velocity correlations in weakly turbulent surface waves

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We derive quantitative predictions for velocity correlations in weakly turbulent surface waves. For capillary surface waves, our theoretical results are shown to be in extremely good agreement with experiment. This provides the strongest evidence so far for the applicability of weak-turbulence theory to capillary waves. For gravity waves the predicted velocity correlations remain to be tested experimentally.
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The Hamiltonian theory of weak turbulence for surface waves has been developed quite extensively [1]. The focus has been centered on the Kolmogorov spectrum $n(k)$, but the experimental attempts to confirm the theory have been very limited. For capillary surface waves, elaborate experimental techniques have been used in order to test the scaling properties derived for the spectrum $n(k)$ [2]. Numerical calculations of the spectrum have also been carried out for capillary waves [3].

In this Brief Report, we consider velocity correlations rather than the spectrum itself. We derive an expression for the variance $\langle|\delta\boldsymbol{v}(R)|^2\rangle$ of the distribution of velocity differences over a (horizontal) distance R. Unlike the spectrum $n(k)$, $\langle|\delta\boldsymbol{v}(R)|^2\rangle$ can rather easily be studied experimentally by measuring the separation of tracers launched on the fluid surface [4,5].

Experiments on relative particle motion have previously been performed for (deep-water) capillary waves [5]. For the weakly turbulent regime, it was argued that

$$\langle|\delta\boldsymbol{v}(R)|^2\rangle \propto 1-b(R/\lambda)^{-1/4}, \tag{1}$$

where λ is the characteristic wavelength of the surface waves, and where b was not determined theoretically. By fitting to experimental data in the turbulent regime, the value of b was estimated to be $b\simeq 0.66$, independent of the forcing f [for $0<\epsilon=(f-f_c)/f_c<0.4$]. Below we give a derivation of Eq. (1), and we quantitatively determine the value of b to be $b=0.649\ldots$, in striking agreement with experiment. To our knowledge, this provides the strongest evidence so far of the applicability of weak turbulence theory to capillary surface waves. Furthermore, we shall derive two expressions similar to Eq. (1) for velocity correlations in gravity waves, and determine the appropriate constants. These expressions provide strong predictions for the velocity correlations in gravity waves.

First, consider deep-water capillary waves. The dispersion relation of these waves is $\omega(k)=(\sigma/\rho)^{1/2}k^{3/2}$, and the isotropic Kolmogorov spectrum is [1,6]

$$n(k)=A(P^2\rho^3/\sigma)^{1/4}k^{-17/4}, \tag{2}$$

where σ is the surface tension coefficient, ρ is the density, and P is the energy flux, which is assumed to be constant. A is a nondimensional constant. We calculate the correlation

$$\langle\boldsymbol{v}(\boldsymbol{r}+\boldsymbol{R})\cdot\boldsymbol{v}(\boldsymbol{r})\rangle=\frac{1}{2\pi}\left\langle\int \boldsymbol{v}_{\boldsymbol{k}}\cdot\boldsymbol{v}_{-\boldsymbol{k}}e^{-i\boldsymbol{k}\cdot\boldsymbol{R}}d\boldsymbol{k}\right\rangle, \tag{3}$$

using the Fourier components

$$\boldsymbol{v}_{\boldsymbol{k}}=2^{-1/2}(\sigma k/\rho^3)^{1/4}\boldsymbol{k}[a_{\boldsymbol{k}}-a^*_{-\boldsymbol{k}}] \tag{4}$$

(the asterisk denotes complex conjugation). $a_{\boldsymbol{k}}$ is the complex wave amplitude entering the canonical equation of motion [1],

$$i\frac{\partial a_{\boldsymbol{k}}}{\partial t}=\frac{\delta H}{\delta a^*_{\boldsymbol{k}}}, \tag{5}$$

where H is the Hamiltonian (the total energy of the fluid). Correlation (3) is related to the velocity-difference variance $\langle|\delta\boldsymbol{v}(R)|^2\rangle$ by the obvious relation

$$\begin{aligned}\langle|\delta\boldsymbol{v}(R)|^2\rangle&=\langle|\boldsymbol{v}(\boldsymbol{r}+\boldsymbol{R})-\boldsymbol{v}(\boldsymbol{r})|^2\rangle\\&=2[\langle|\boldsymbol{v}(\boldsymbol{r})|^2\rangle-\langle\boldsymbol{v}(\boldsymbol{r}+\boldsymbol{R})\cdot\boldsymbol{v}(\boldsymbol{r})\rangle].\end{aligned} \tag{6}$$

When no nonlinearities are present, the correlator $\langle a_{\boldsymbol{k}}a_{-\boldsymbol{k}}\rangle$ in Eq. (3) vanishes. In the present system with only weak nonlinearities, we shall therefore assume that $\langle a_{\boldsymbol{k}}a_{-\boldsymbol{k}}\rangle$ is negligibly small compared to the isotropic spectrum $n(k)=\langle a_{\boldsymbol{k}}a^*_{\boldsymbol{k}}\rangle$. Hence

$$\langle\boldsymbol{v}(\boldsymbol{r}+\boldsymbol{R})\cdot\boldsymbol{v}(\boldsymbol{r})\rangle=\frac{1}{2\pi}(\sigma/\rho^3)^{1/2}\int k^{5/2}n(k)e^{-i\boldsymbol{k}\cdot\boldsymbol{R}}d\boldsymbol{k}. \tag{7}$$

For the velocity-difference variance, we now have

$$\begin{aligned}\langle|\delta\boldsymbol{v}(R)|^2\rangle&=\frac{1}{\pi}(\sigma/\rho^3)^{1/2}\int k^{5/2}n(k)(1-e^{-i\boldsymbol{k}\cdot\boldsymbol{R}})d\boldsymbol{k}\\&=\frac{1}{\pi}(\sigma/\rho^3)^{1/2}\int_{k_{\min}}^{k_{\max}}k^{7/2}n(k)\\&\quad\times\left[\int_0^{2\pi}(1-e^{-ikR\cos\theta})d\theta\right]dk,\end{aligned} \tag{8}$$

where $k_{\min}$, which is related to the inverse system size, is set to zero, and $k_{\max}$ is set to $2\pi/\lambda$. At length scales below the characteristic wavelength λ, the nature of the fluid motion differs from that at larger length scales [7]. Accordingly, we do not consider wave numbers larger than $k_{\max}$, and we only apply the theoretical results obtained here to length scales above λ.

The angular integral in Eq. (8) is expressed via the zeroth Bessel function J_0, and thus Eq. (8) becomes

$$\langle|\delta\boldsymbol{v}(R)|^2\rangle = 2(\sigma/\rho^3)^{1/2}\int_0^{2\pi/\lambda} k^{7/2} n(k)[1-J_0(kR)]dk$$
$$= 2A(P^2\sigma/\rho^3)^{1/4}R^{-1/4}\int_0^{2\pi R/\lambda} s^{-3/4}\times[1-J_0(s)]ds, \quad (9)$$

where $s=Rk$, and $n(k)$ from Eq. (2) has been inserted.

For the Bessel function integral, we find [8]

$$\int_0^{2\pi R/\lambda} s^{-3/4}J_0(s)ds = 2^{-3/4}\frac{\Gamma(1/8)}{\Gamma(7/8)} - \frac{2\pi R}{\lambda}\left\{\frac{7}{4}J_0\left(\frac{2\pi R}{\lambda}\right)\mathcal{S}_{-7/4,-1}\left(\frac{2\pi R}{\lambda}\right) + J_1\left(\frac{2\pi R}{\lambda}\right)\mathcal{S}_{-3/4,0}\left(\frac{2\pi R}{\lambda}\right)\right\}$$
$$= 2^{-3/4}\frac{\Gamma(1/8)}{\Gamma(7/8)} + \mathcal{O}((\lambda/R)^{5/4}), \quad (10)$$

for $R>\lambda$. Here Γ is the gamma function and $\mathcal{S}_{\mu,\nu}$ are the Lommel functions. The $\mathcal{O}((\lambda/R)^{5/4})$ correction is of oscillatory nature, and contributes for any $R>\lambda$ numerically very little to the expression (at worst less than 2%). The correction can therefore be neglected. For the variance of the velocity difference, we finally obtain

$$\langle|\delta\boldsymbol{v}(R)|^2\rangle = 2A\left(\frac{P^2\sigma}{\rho^3}\right)^{1/4}R^{-1/4}\left[4\left(\frac{2\pi R}{\lambda}\right)^{1/4} - 2^{-3/4}\frac{\Gamma(1/8)}{\Gamma(7/8)}\right]$$
$$= 8A\left(\frac{2\pi P^2\sigma}{\lambda\rho^3}\right)^{1/4}\left[1-b\left(\frac{R}{\lambda}\right)^{-1/4}\right], \quad (11)$$

where

$$b = \frac{\Gamma(1/8)}{8\pi^{1/4}\Gamma(7/8)} = 0.649\ldots. \quad (12)$$

As already mentioned, this value of b is in strikingly good agreement with the experimentally found value $b=0.66$ [5]. Not only does the experimental data fit the previously predicted form of Eq. (11), by finding the numerical value of b in (11) we have now also shown that the agreement is quantitatively correct. This strongly supports the applicability of weak turbulence theory to the observed turbulence in capillary waves. We note that in Eq. (11) only P depends on the forcing.

In the experiments, the waves were driven by different forcings f corresponding to the drives $\epsilon=(f-f_c)/f_c$. For the two different drives $\epsilon_1=0.13$ and $\epsilon_2=0.24$, we find $\langle|\delta\boldsymbol{v}(R)|^2\rangle_{\epsilon_2}/\langle|\delta\boldsymbol{v}(R)|^2\rangle_{\epsilon_1}\simeq 1.7$. Hence, according to Eq. (11), the ratio of the energy fluxes is $P(\epsilon_2)/P(\epsilon_1)\simeq 2.9$. From perturbation theory in small ϵ, it is known that $P(\epsilon)\propto\sigma\gamma(\epsilon+c\epsilon^2+\cdots)$ where $\gamma=2\nu k^2$ is the linear bulk damping, ν is the kinematic viscosity, and c is a nondimensional constant. From the values $\epsilon_2/\epsilon_1=1.8$ and $(\epsilon_2/\epsilon_1)^2=3.4$, an energy flux ratio of 2.9 shows, as expected, that nonlinear damping is important in the weakly turbulent regime.

For gravity waves, a formula for $\langle|\delta\boldsymbol{v}(R)|^2\rangle$ similar to that for capillary waves [Eq. (11)] can be derived. The dispersion relation is $\omega(k)=(gk)^{1/2}$, where g is the gravitation constant. Moreover, because $\omega\propto k^\alpha$ with $\alpha<1$, there are not one but two isotropic Kolmogorov spectra. One spectrum corresponds to a constant energy flux P (energy conserved) [1,9], as for capillary waves,

$$n(k) = A_1(P\rho^2)^{1/3}k^{-4}. \quad (13)$$

The other spectrum is associated with a constant flux of wave action Q (number of ‘‘quasiparticles’’ $\int n(k)d\boldsymbol{k}$ conserved) [1,10],

$$n(k) = A_2(Q^2 g\rho^4)^{1/6}k^{-23/6}. \quad (14)$$

The expression for the Fourier transform of the velocity [Eq. (4)] is replaced by $\boldsymbol{v}_{\boldsymbol{k}} = 2^{-1/2}(g/\rho^2 k)^{1/4}\boldsymbol{k}[a_{\boldsymbol{k}} - a^*_{-\boldsymbol{k}}]$, and instead of Eq. (9) we obtain

$$\langle|\delta\boldsymbol{v}(R)|^2\rangle = 2(g^{1/2}/\rho)\int_0^{2\pi/\lambda} k^{5/2}n(k)[1-J_0(kR)]dk. \quad (15)$$

An integration by parts followed by a calculation similar to that for capillary waves for a constant energy flux then yields

$$\langle|\delta\boldsymbol{v}(R)|^2\rangle = 2A_1 g^{1/2}\left(\frac{P}{\rho}\right)^{1/3}R^{1/2}\int_0^{2\pi R/\lambda} s^{-3/2}[1-J_0(s)]ds$$
$$= 4A_1\left(\frac{g\lambda}{2\pi}\right)^{1/2}\left(\frac{P}{\rho}\right)^{1/3}\left[-1+b_1\left(\frac{R}{\lambda}\right)^{1/2} - \left(\frac{2\pi R}{\lambda}\right)^{3/2}\left\{\frac{1}{2}J_1\left(\frac{2\pi R}{\lambda}\right)\mathcal{S}_{-3/2,0}\left(\frac{2\pi R}{\lambda}\right) + J_0\left(\frac{2\pi R}{\lambda}\right)\left[\mathcal{S}_{-1/2,1}\left(\frac{2\pi R}{\lambda}\right) - \left(\frac{2\pi R}{\lambda}\right)^{-3/2}\right]\right\}\right]$$
$$= 4A_1\left(\frac{g\lambda}{2\pi}\right)^{1/2}\left(\frac{P}{\rho}\right)^{1/3}\left[-1+b_1\left(\frac{R}{\lambda}\right)^{1/2} + \mathcal{O}((\lambda/R)^{3/2})\right], \quad (16)$$

with

$$b_1 = \frac{\pi^{1/2}\Gamma(3/4)}{\Gamma(5/4)} = 2.396\ldots. \quad (17)$$

For a constant flux of wave action, we obtain

$$\langle|\delta\boldsymbol{v}(R)|^2\rangle = 2A_2\left(\frac{Qg^2}{\rho}\right)^{1/3} R^{1/3}\int_0^{2\pi R/\lambda} s^{-4/3}[1-J_0(s)]ds$$
$$= 6A_2\left(\frac{Qg^2\lambda}{2\pi\rho}\right)^{1/3}\left[-1+b_2\left(\frac{R}{\lambda}\right)^{1/3} - \left(\frac{2\pi R}{\lambda}\right)^{4/3}\left\{\frac{1}{3}J_1\left(\frac{2\pi R}{\lambda}\right)\ \mathcal{S}_{-4/3,0}\left(\frac{2\pi R}{\lambda}\right) + J_0\left(\frac{2\pi R}{\lambda}\right)\left[\mathcal{S}_{-1/3,1}\left(\frac{2\pi R}{\lambda}\right) - \left(\frac{2\pi R}{\lambda}\right)^{-4/3}\right]\right\}\right]$$
$$= 6A_2\left(\frac{Qg^2\lambda}{2\pi\rho}\right)^{1/3}\left[-1+b_2\left(\frac{R}{\lambda}\right)^{1/3} + \mathcal{O}((\lambda/R)^{4/3})\right], \tag{18}$$

with

$$b_2 = \frac{\pi^{1/3}\Gamma(5/6)}{\Gamma(7/6)} = 1.782\ldots. \tag{19}$$

Again, the corrections in Eqs. (16) and (18) are of oscillatory nature, and account for less than 2% of the total value for any $R > \lambda$. The predictions given by Eqs. (16) and (17) and (18) and (19) for $\langle|\delta\boldsymbol{v}(R)|^2\rangle$ in gravity waves have not yet been experimentally tested.

In conclusion, we have derived quantitative predictions for velocity correlations in weakly turbulent surface waves. For (deep-water) capillary waves our results, compared to available experimental data, provide strong evidence for the applicability of weak turbulence theory. For gravity waves corresponding experimental data are not yet available to test our predictions.

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